

GODEAUX SURFACES WITH AN ENRIQUES INVOLUTION AND SOME STABLE DEGENERATIONS

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ABSTRACT. We give an explicit description of the Godeaux surfaces S (minimal surfaces of general type with $K_S^2 = \chi(\mathcal{O}_S) = 1$) that admit an involution σ such that S/σ is birational to an Enriques surface; these surfaces give a 6-dimensional unirational irreducible subset of the moduli space of surfaces of general type.

In addition, we describe the Enriques surfaces that are birational to the quotient of a Godeaux surface by an involution and we show that they give a 5-dimensional unirational irreducible subset of the moduli space of Enriques surfaces.

Finally, by degenerating our description we obtain some examples of non-normal stable Godeaux surfaces; in particular we show that one of the families of stable Gorenstein Godeaux surfaces classified in [FPR] consists of smoothable surfaces.

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1. INTRODUCTION

A Godeaux surface is (the canonical model of) a minimal surface of general type with $K_S^2 = \chi(\mathcal{O}_S) = 1$. These surfaces have been intensely studied since the 1970's, but a complete classification is still lacking. A very synthetic summary of the state of the art is as follows:

- the algebraic fundamental group π_1^{alg} of a Godeaux surface is cyclic of order ≤ 5 ([Miy75]); in particular if S is a Godeaux surface then π_1^{alg} is abelian and thus it coincides with the torsion subgroup $\text{Tors}(S)$ of $\text{Pic}(S)$;
- the Godeaux surfaces with π_1^{alg} of order 3, 4, 5 are explicitly described; to each of these possibilities for π_1^{alg} there corresponds an irreducible unirational 8-dimensional connected component of the moduli space ([Rei78]);
- Godeaux surfaces with $\pi_1^{\text{alg}} = 0$ or \mathbb{Z}_2 do exist, but little is known about the geometry of the moduli space ([Bar84], [Bar85], [LP07], [PPS13]).

One of the strategies to overcome the difficulties of the classification is to restrict one's attention to a subclass of Godeaux surfaces with an extra structure, for instance those admitting an involution. This has been done by Keum-Lee ([KL00]) and by Calabri, Ciliberto and Mendes Lopes ([CCML07]), who described the possibilities for the quotient surface and the fixed locus of the involution.

Here we study in detail the case when the quotient surface is birational to an Enriques surface (in this case, we call σ an “Enriques involution”). Since in this case $\text{Tors}(S) \cong \mathbb{Z}_4$ ([CCML07]), the universal cover \tilde{S} of the Godeaux surface is a complete intersection in a weighted projective space ([Rei78]). The involution σ lifts to an involution $\tilde{\sigma}$ of \tilde{S} and the action of $\tilde{\sigma}$ on the canonical ring of \tilde{S} can be determined by means of a careful study of linear systems on the quotient Enriques surface, yielding the classification (Theorem 3.2). As a consequence, the locus of Godeaux surfaces with an Enriques involution is irreducible of dimension 6 (Corollary 3.3) and the locus of Enriques surfaces that are birational to the quotient of a Godeaux surface by an involution (Enriques surfaces “of Godeaux-quotient type”) is irreducible of dimension 5 (Corollary 4.2). In §4 we specialize a classical construction of the special Enriques surfaces ([Hor78a], [Hor78b]) to obtain Enriques surfaces of Godeaux-quotient type: since our construction depends on 5 parameters, by the irreducibility of the locus of Enriques surfaces of Godeaux-quotient type it gives the general Enriques surface of Godeaux-quotient type.

The moduli space of (canonical models) of surfaces of general type can be compactified by considering a larger class of surfaces, the so-called stable surfaces (cf. §6 for the definition). The stable Gorenstein surfaces with $K^2 = 1$ (thus including the stable Gorenstein Godeaux surfaces) are investigated

in the series of recent papers [FPR14b], [FPR14a], [FPR]. In §6 we give an explicit construction of the general Godeaux surface with an Enriques involution and use it to produce stable Godeaux surfaces. In this way we produce a normal Gorenstein degeneration with an elliptic singularity of degree 4, whose existence was predicted in [FPR14b], and we show the smoothability of one of the families of non-normal Godeaux surfaces with normalization isomorphic to \mathbb{P}^2 ([FPR14b], [FPR]). In addition we give examples of stable non-normal Godeaux surfaces with Cartier index equal to 2 whose normalization is not ruled, thus showing that the main result of [FPR14b] does not hold without the Gorenstein assumption.

Finally, a remark on the methods: the constructions of the general Enriques surface of Godeaux-quotient type (§4) and of the general Godeaux surface with and Enriques involution (§6) are based:

- (a) on the fact that, for a certain involution τ of Y and for a certain double/bidouble cover $p: X \rightarrow Y$, τ can be lifted to an involution of X ;
- (b) on the fact that the 2-divisibility of p^*D for a certain divisor D on Y implies that D is also 2-divisible.

The conditions under which (a) and (b) above hold for a general bidouble cover are investigated in §3: we believe that this section is of independent interest.

Acknowledgments: we are grateful to the editors of this volume for inviting us to contribute to it. We hope that, although the topic is not directly related to the work of Corrado Segre, the influence of the classical Italian tradition of algebraic geometry that pervades the paper makes it a suitable addition to this project.

Notation and conventions: We work over the complex numbers. Following the terminology of [Kol13], a variety is called *demi-normal* if it satisfies condition S_2 of Serre and in codimension 1 it is either smooth or double crossings. If X is a demi-normal projective variety, then the dualizing sheaf ω_X is divisorial; we denote by K_X a canonical divisor, that is, a Weil divisor such that $\mathcal{O}_X(K_X) \cong \omega_X$. For a projective variety X we denote by $\text{Tors}(X)$ the torsion subgroup of $\text{Pic}(X)$ and by $\text{Pic}(X)[d]$ the subgroup consisting of the d -torsion elements. We use \equiv to denote linear equivalence of divisors and \sim to denote numerical equivalence of \mathbb{Q} -divisors.

Throughout all the paper G is used to denote the Galois group of a finite cover.

2. GALOIS COVERS AND DIVISIBILITY

In this section we first summarize the theory of [Par91] and [AP12] for covers with Galois group \mathbb{Z}_2 and \mathbb{Z}_2^2 ; the need to cover also the case of non-normal covers arises because in §6 we consider stable Godeaux surfaces.

Then we present some general results on liftability of automorphisms to double and bidouble covers that are needed in the rest of the paper. Although these results are probably known to experts, to our knowledge they have not been written down elsewhere and we believe that they are of independent interest.

2.1. Double and bidouble covers. Let G be a finite group. A G -cover is a finite map of algebraic varieties $f: X \rightarrow Y$ that is the quotient map for a generically faithful G -action, namely such that for every component Y_i of Y the G -action on the restricted cover $X \times_Y Y_i \rightarrow Y_i$ is faithful. The cover is *abelian* if G is an abelian group: for the general theory of abelian covers we refer the reader to [Par91] for the case X normal and Y smooth and to [AP12] for a more general treatment.

Here we are mainly interested in the case $G \cong \mathbb{Z}_2$ (“double covers”) and $G \cong \mathbb{Z}_2^2$ (“bidouble covers”); for simplicity, we assume throughout that $H^0(Y, \mathcal{O}_Y) = \mathbb{C}$.

Assume first that $f: X \rightarrow Y$ is an abelian cover with group G such that X is normal and Y is smooth. Then f is flat and the branch locus is a divisor; we denote by B the branch divisor with reduced structure. For $G = \mathbb{Z}_2$, we have $f_*\mathcal{O}_X = \mathcal{O}_Y \oplus L^{-1}$, where L is a line bundle, G acts on L^{-1} as multiplication by -1 and the multiplication map $L^{-1} \otimes L^{-1} \rightarrow \mathcal{O}_Y$ induces an isomorphism $L^{\otimes 2} \cong \mathcal{O}_Y(B)$. The pair (L, B) is called the *building data* of the double cover and it determines $f: X \rightarrow Y$ uniquely up to isomorphism of covers, since we assume $H^0(\mathcal{O}_Y) = \mathbb{C}$. We say for short that $f: X \rightarrow Y$ is the double cover given by the equivalence relation $2L \equiv B$.

One can reverse this construction: given building data (L, B) , i.e. given an effective divisor B and a line bundle L satisfying the relation $2L \equiv B$, one can choose an isomorphism $\phi: L^{\otimes 2} \rightarrow \mathcal{O}_Y(B)$, use it to define an associative multiplication on $\mathcal{O}_Y \oplus L^{-1}$, set $X := \text{Spec}(\mathcal{O}_Y \oplus L^{-1})$ and take f to be the natural map $X \rightarrow Y$. This construction makes sense more generally for any effective Cartier divisor B (not necessarily reduced) and line bundle L such that $2L \equiv B$ on an arbitrary variety Y . The flat double cover $f: X \rightarrow Y$ is called the *standard cover* associated with (L, B) ; it is not hard to show that every flat double cover is obtained this way, i.e., it is standard.

The situation is similar for bidouble covers. We start again by considering the case X normal and Y smooth. We write χ_1, χ_2, χ_3 for the three non-trivial characters of $G \cong \mathbb{Z}_2^2$ and denote by $g_i \in G$ the generator of $\ker \chi_i$. The branch divisor B decomposes as $B = B_1 + B_2 + B_3$, where B_i is the image of the divisorial part of the fixed locus of g_i and we have a splitting $f_*\mathcal{O}_X = \mathcal{O}_Y \oplus L_1^{-1} \oplus L_2^{-1} \oplus L_3^{-1}$, where G acts on L_i^{-1} as multiplication by the character χ_i . As in the case of double covers, the multiplication in $f_*\mathcal{O}_X$ induces isomorphisms, and therefore equivalence relations:

$$(2.1) \quad 2L_i \equiv B_j + B_k, \quad L_i + L_j \equiv L_k + B_k,$$

where (i, j, k) is a permutation of $(1, 2, 3)$. Again, (L_i, B_i) , $i = 1, 2, 3$, are called the building data of the bidouble cover and determine $f: X \rightarrow Y$ up to isomorphism of \mathbb{Z}_2^2 -covers. It is easy to see that (2.1) is equivalent to the smaller set of equations:

$$(2.2) \quad 2L_1 \equiv B_2 + B_3, \quad 2L_2 \equiv B_1 + B_3, \quad L_3 \equiv L_1 + L_2 - B_3,$$

and in particular L_3 can be recovered from the remaining data. We call $(L_1, L_2, B_1, B_2, B_3)$ the *reduced building data* and we say for short that the cover is given by the relations $2L_1 \equiv B_2 + B_3$, $2L_2 \equiv B_1 + B_3$.

As in the case of double covers, we can perform the reverse construction in greater generality, starting with line bundles L_1, L_2 and effective Cartier divisors satisfying (2.2), and obtain a *standard bidouble cover* of an arbitrary variety Y . Again, the building data determine the standard cover uniquely up to isomorphism of bidouble covers, since we assume $H^0(\mathcal{O}_Y) = \mathbb{C}$. We set $B = B_1 + B_2 + B_3$; observe that B may be non-reduced. We recall the following:

Proposition 2.1 ([AP12], Cor. 1.10). *Let $f: X \rightarrow Y$ be a double or bidouble cover with Y smooth and X demi-normal. Then f is a standard cover and every component of B has multiplicity at most 2.*

2.2. Lifting automorphisms to double and bidouble covers. We discuss in detail the case of bidouble covers; the case of double covers can be treated by similar, but simpler, arguments.

Let Y be a variety with $H^0(Y, \mathcal{O}_Y) = \mathbb{C}$, let $f: X \rightarrow Y$ be a standard bidouble cover given by relations $2L_1 \equiv B_2 + B_3$ and $2L_2 \equiv B_1 + B_3$ and denote by $G \cong \mathbb{Z}_2^2$ the Galois group of f . Let $\rho \in \text{Aut}(Y)$ be an automorphism such that one of the following holds:

- (a) $\rho^* B_i = B_i$, $i = 1, 2, 3$, and $\rho^* L_j \equiv L_j$, $j = 1, 2$
- (b) $\rho^* B_1 = B_2$, $\rho^* B_2 = B_1$, $\rho^* B_3 = B_3$, $\rho^* L_1 \equiv L_2$, $\rho^* L_2 \equiv L_1$.

In either case, the automorphism ρ lifts to an automorphism $\tilde{\rho}$ of X . Indeed, consider the following cartesian diagram:

$$(2.3) \quad \begin{array}{ccc} X' & \xrightarrow{\rho'} & X \\ f' \downarrow & & \downarrow f \\ Y & \xrightarrow{\rho} & Y. \end{array}$$

In case (a), f' is a standard bidouble cover given by the same building data as f , hence it is isomorphic to f via an isomorphism compatible with the action of $G \cong \mathbb{Z}_2^2$ and $\tilde{\rho}$ is obtained by composing such an isomorphism with ρ' ; in case (b) we modify the G -action on X' by composing with the automorphism of G that switches g_1 and g_2 and argue as in case (a).

Let \tilde{G} be the subgroup of $\text{Aut}(X)$ generated by G and by $\tilde{\rho}$. Then there is a short exact sequence of groups:

$$1 \rightarrow G \rightarrow \tilde{G} \rightarrow \langle \rho \rangle \rightarrow 1.$$

The group \tilde{G} is abelian in case (a), since $\tilde{\rho}$ preserves the decomposition of $f_*\mathcal{O}_X$ into G -eigensheaves, and it is non abelian in case (b); in particular, if $\rho^2 = 1$ then $\tilde{\rho}^4 = 1$ and, by the classification of groups of order 8, \tilde{G} is isomorphic either to \mathbb{Z}_2^3 or $\mathbb{Z}_2 \times \mathbb{Z}_4$ in case (a) and to the dihedral group D_4 in case (b).

In the case of double covers one assumes that $\rho^*B = B$ and $\rho^*L \equiv L$: in this case $\tilde{\rho}$ commutes with the action of $G \cong \mathbb{Z}_2$ and the group \tilde{G} is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_d$ or to \mathbb{Z}_{2d} .

2.3. Divisibility. Recall that a Cartier divisor or line bundle on a projective variety is said to be *even* if its class is divisible by 2 in $\text{Pic}(X)$.

Lemma 2.2. *Let $f: X \rightarrow Y$ be a cyclic étale cover of projective varieties and let K be the kernel of $f^*: \text{Pic}(Y) \rightarrow \text{Pic}(X)$. Let D be a Cartier divisor on Y such that f^*D is even.*

If $\text{Pic}(X)[2] = 0$, then the class of D is divisible by 2 in $\text{Pic}(Y)/K$.

Proof. Let $\tilde{M} \in \text{Pic}(X)$ be a line bundle such that $2\tilde{M} \equiv f^*D$. Denote by g a generator of the Galois group G of f ; since D is g -invariant, we have $2g^*\tilde{M} \equiv f^*D \equiv 2\tilde{M}$. Since $\text{Pic}(X)[2] = 0$ it follows that the line bundles \tilde{M} and $g^*\tilde{M}$ are isomorphic and therefore \tilde{M} admits a G -linearization (G is cyclic). Since f is étale, \tilde{M} descends to a line bundle M on Y . One has $f^*(2M - D) \equiv 0$, hence $D = 2M$ in $\text{Pic}(Y)/K$. \square

Lemma 2.3. *Let $f: X \rightarrow Y$ be a cyclic étale cover of degree d of projective varieties and let D be an effective Cartier divisor on Y such that f^*D is even. Assume that $\text{Pic}(X)[2] = 0$ and denote by $h: Z \rightarrow X$ the flat double cover branched on f^*D . Then:*

- (i) *the composite map $f \circ h: Z \rightarrow Y$ is a Galois cover with Galois group \tilde{G} isomorphic to \mathbb{Z}_{2d} or to $\mathbb{Z}_2 \times \mathbb{Z}_d$;*
- (ii) *D is even iff \tilde{G} is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_d$.*

Notice that Lemma 2.3 is interesting only if d is even. Indeed, if d is odd then $\tilde{G} \cong \mathbb{Z}_{2d} \cong \mathbb{Z}_2 \times \mathbb{Z}_d$ is cyclic and statement (ii) just says that D is even, as we already know by Lemma 2.2.

Proof. (i) Let $\tilde{L} \in \text{Pic}(X)$ be the only element such that $2\tilde{L} \equiv f^*D$. Let g be a generator of the Galois group G of f ; by construction f^*D is G -invariant, hence arguing as in the proof of Lemma 2.2 one sees that \tilde{L} is also G -invariant. Therefore by the discussion of §2.2 it is possible to lift g to an automorphism \tilde{g} of Z and the subgroup \tilde{G} of $\text{Aut}(Z)$ generated by \tilde{g} and by the involution ι associated with h is isomorphic to \mathbb{Z}_{2d} or $\mathbb{Z}_2 \times \mathbb{Z}_d$. The former case occurs iff \tilde{G} is generated by \tilde{g} or by $\tilde{g}\iota$. Clearly, \tilde{G} is the Galois group of $f \circ h$.

(ii) Assume that $\tilde{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_d$ and let \tilde{g} be an element of order d that lifts g : then \tilde{g} acts freely on Z by construction and $Z/\tilde{g} \rightarrow Y$ is a flat

double cover. Since $Z \rightarrow Z/\tilde{g}$ is étale, it is easy to see that $Z/\tilde{g} \rightarrow Y$ is standard with building data (L, D) , for some $L \in \text{Pic}(Y)$, hence D is even. Conversely, assume that D is even and let $L \in \text{Pic}(Y)$ be such that $2L \equiv D$. We have $f^*L = \tilde{L}$ since $\text{Pic}(X)[2] = 0$ and therefore $Z \rightarrow Y$ is the fiber product of $f: X \rightarrow Y$ and of the double cover given by the relation $2L \equiv D$ and has Galois group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_d$. \square

Let X be a surface and let $p_1, \dots, p_k \in X$ be A_1 singularities (“nodes”). We say that p_1, \dots, p_k is an *even set of nodes* of X if there exists a double cover of X branched precisely on p_1, \dots, p_k . Denote by $X' \rightarrow X$ the minimal resolution of the singularities p_1, \dots, p_k and by C_i the exceptional curve over p_i ; C_i is a *nodal curve*, i.e., it is smooth rational and $C_i^2 = -2$. The set $\{p_1, \dots, p_k\}$ is even if and only if $C_1 + \dots + C_k$ is an even divisor of X' . By using the adjunction formula on X' it is easy to check that an even set of nodes has cardinality divisible by 4.

Lemma 2.4. *Let Y be a smooth projective surface, let B_1, B_2 be even curves of Y meeting transversely at smooth points q_1, \dots, q_k of Y .*

If $f: X \rightarrow Y$ is a flat double cover branched on $B := B_1 + B_2$, then the points p_1, \dots, p_k lying above q_1, \dots, q_k are an even set of nodes of X .

Proof. The fact that p_1, \dots, p_k are nodes of X can be checked easily by a local computation. Let $L_3 \in \text{Pic}(X)$ be such that $f_*\mathcal{O}_X = \mathcal{O}_Y \oplus L_3^{-1}$, so that f is given by the relation $2L_3 \equiv B$. Choose $L_1 \in \text{Pic}(X)$ with $2L_1 \equiv B_2$ and set $L_2 := L_3 - L_1$. As explained in §2.1, the relations $2L_1 \equiv B_2$ and $2L_2 \equiv B_1$ determine a standard bidouble cover $h: Z \rightarrow Y$ (we take $B_3 = 0$). For $i = 1, 2$ denote by $g_i \in G \cong \mathbb{Z}_2^2$ the element that fixes $h^{-1}B_i$ pointwise and set $g_3 = g_1 + g_2$. Then Z/g_3 is isomorphic to X and the quotient map $Z \rightarrow Z/g_3$ is a double cover branched precisely on p_1, \dots, p_k . \square

3. GODEAUX SURFACES WITH AN ENRIQUES INVOLUTION

In this section we study the following situation:

- S is a numerical Godeaux surface, i.e., a smooth minimal surface of general type with $K_S^2 = 1$ and $p_g(S) = q(S) = 0$
- $\sigma \in \text{Aut}(S)$ is an involution such that $\Sigma := S/\sigma$ is birational to an Enriques surface.

We call the involution σ an *Enriques involution*. Godeaux surfaces with an involution have been studied in [KL00] and in [CCML07]; in particular, in [CCML07] it is proven that a Godeaux surface S with an Enriques involution has $\text{Tors}(S) \cong \mathbb{Z}_4$. In addition, the possible automorphism groups of numerical Godeaux surfaces with torsion of order ≥ 3 have been listed in [Mag], but without analyzing the quotient surfaces.

We recall the following example [KL00, Ex. 4.3]:

Example 3.1. Let S be a Godeaux surface with $\text{Tors}(S) \cong \mathbb{Z}_4$ and let $\tilde{S} \rightarrow S$ be the universal cover, i.e. the degree 4 cyclic cover given by

$\text{Tors}(S)$. By [Rei78], the minimal model \tilde{S}_{can} of \tilde{S} is canonically embedded in $\mathbb{P}(1, 1, 1, 2, 2)$, with coordinates x_1, x_2, x_3, y_1, y_3 , as the zero locus of two homogeneous equations q_0 and q_2 of degree 4.

The equation q_0 involves the monomials:

$$x_1^4, x_2^4, x_3^4, x_1^2 x_3^2, x_1 x_3 x_2^2, x_1 x_2 y_1, x_2 x_3 y_3, y_1 y_3,$$

and q_2 involves the monomials:

$$x_1^2 x_2^2, x_2^2 x_3^2, x_1^3 x_3, x_1 x_3^3, x_1 x_2 y_3, x_2 x_3 y_1, y_1^2, y_3^2.$$

We denote by $G \cong \mathbb{Z}_4$ the Galois group of $\tilde{S} \rightarrow S$: the group G acts freely also on \tilde{S}_{can} and the quotient surface is the canonical model S_{can} of S . The action of G extends to the ambient $\mathbb{P}(1, 1, 1, 2, 2)$ and there is a generator $g \in G$ that acts by $(x_1, x_2, x_3, y_1, y_3) \mapsto (ix_1, -x_2, -ix_3, iy_1, -iy_3)$.

Now we define an involution $\tilde{\sigma}$ of $\mathbb{P}(1, 1, 1, 2, 2)$ by $(x_1, x_2, x_3, y_1, y_3) \mapsto (-x_1, x_2, -x_3, y_1, y_3)$; the involution $\tilde{\sigma}$ commutes with g . We assume from now on that the polynomial q_0 does not involve $x_1 x_2 y_1$, $x_2 x_3 y_3$ and the polynomial q_2 does not involve $x_2 x_3 y_1$, $x_1 x_2 y_3$, so that q_0 and q_2 are invariant under $\tilde{\sigma}$. Hence $\tilde{\sigma}$ acts on \tilde{S}_{can} and descends to an involution σ of S_{can} and of its minimal resolution S .

The divisorial part R of the fixed locus σ on S_{can} is the paracanonical curve defined by $x_2 = 0$, hence it is a connected curve of genus 2; if S_{can} is smooth then R is also smooth, and by Cor. 4.8 and Prop. 7.10 of [CCML07] it follows that σ is an Enriques involution. Since the quotient of a smooth surface by an involution has canonical singularities, it follows that for every smooth S_{can} as above the involution σ of S is an Enriques involution. Using Bertini's theorem, it is not difficult to see that if q_0 and q_2 are general the surface $S_{\text{can}} = S$ is smooth.

In this section we characterize the quotient surface S/σ and, exploiting this characterization, we prove the following classification results:

Theorem 3.2. *Let S be a Godeaux surface and let $\sigma \in \text{Aut}(S)$ be an Enriques involution.*

Then S is as in Example 3.1.

The surfaces in Example 3.1 correspond to case R_1 of Table 2 of [Mag], hence they form an irreducible unirational subset of dimension 6 of the moduli space of Godeaux surfaces with torsion of order 4. Hence Theorem 3.2 yields immediately:

Corollary 3.3. *The Godeaux surfaces with an Enriques involution give an irreducible unirational subset \mathcal{GE} of dimension 6 of the moduli space of Godeaux surfaces with torsion of order 4.*

A possible strategy for proving Theorem 3.2 would be to use the description given in [Mag] of the Godeaux surfaces with torsion of order 4 that admit an involution and decide which involutions are Enriques by looking at the fixed locus, as we have done in Example 3.1. However we prefer to

use a more conceptual approach, based on a detailed study of linear systems on the quotient Enriques surfaces, that gives also a description of the family of such Enriques surfaces (cf. §4).

The rest of the section is devoted to proving Theorem 3.2; we start by fixing some notation.

We denote by $\pi: S \rightarrow \Sigma$ the quotient map; by [CCML07, Prop. 4.5], the bicanonical map of S is composed with σ and $\text{Fix}(\sigma)$ consists of a smooth curve R and of 5 isolated fixed points p_1, \dots, p_5 . We set $q_i = \pi(p_i)$, $i = 1, \dots, 5$ and $B := \pi(R)$. There is a commutative diagram

$$(3.1) \quad \begin{array}{ccc} V & \xrightarrow{\epsilon} & S \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ W & \xrightarrow{\eta} & \Sigma \end{array}$$

where ϵ is the blow up of S at p_1, \dots, p_5 , η is the minimal resolution of Σ and $\tilde{\pi}$ is a flat double cover. For $i = 1, \dots, 5$ we denote by C_i the exceptional curve over q_i ; the C_i are nodal curves, that is, they are smooth rational and $C_i^2 = -2$. By [CCML07, Prop. 3.9] and Lemma 4.11, ibidem, there exists a birational morphism $f: W \rightarrow Y$ such that:

- Y is a smooth Enriques surface
- the exceptional locus of f is disjoint from the C_i
- there is a flat double cover $p: X \rightarrow Y$ fitting in the commutative diagram:

$$(3.2) \quad \begin{array}{ccccc} X & \xleftarrow{g} & V & \xrightarrow{\epsilon} & S \\ p \downarrow & & \tilde{\pi} \downarrow & & \downarrow \pi \\ Y & \xleftarrow{f} & W & \xrightarrow{\eta} & \Sigma \end{array}$$

where X has canonical singularities and g is the minimal resolution.

Also, we abuse notation and we denote by the same letter a curve in V , resp. W , and its image in X , resp. Y . This should not be confusing for the reader, since we will mostly work with the cover $p: X \rightarrow Y$ and forget about $\tilde{\pi}: V \rightarrow W$. The branch curve $B \subset Y$ has at most negligible singularities and it is disjoint from C_1, \dots, C_5 ; the flat cover p is given by the linear equivalence $2L \equiv B + C_1 + \dots + C_5$. For $i = 1, \dots, 5$, the surface X is smooth above the curve C_i and $p^*C_i = 2\Gamma_i$, with Γ_i a -1 -curve. By contracting $\Gamma_1, \dots, \Gamma_5 \subset X$, one obtains an intermediate object between the minimal surface S and its canonical model S_{can} ; in particular p^*B is the pull back of $2K_{S_{\text{can}}}$, hence B is nef and $B^2 = 2$. Since $h^i(B) = h^i(K_Y + (K_Y + B))$, by Kawamata-Viehweg vanishing we have $h^i(B) = 0$ for $i > 0$, so $h^0(B) = 2$. We have $L^2 = -2$, hence $\chi(L) = 0$. Since $h^i(L) = 0$ for $i > 0$ by Kawamata-Viehweg vanishing, we have $h^0(L) = 0$ as well.

Recall (cf. [CD89]) that an *elliptic half-pencil* of an Enriques surface Y is an effective divisor E such that $|2E|$ is a free pencil of elliptic curves of Y . One has:

Proposition 3.4. *In the above setting, up to reordering C_1, \dots, C_5 , we have:*

- (i) *there exists an elliptic half-pencil E of Y such that $B \in |2E + C_5 + K_Y|$;*
- (ii) *the divisor $K_Y + C_1 + \dots + C_4$ is divisible by 2 in $\text{Pic}(Y)$.*

Proof. (i) Let $D \in |B|$ be general. By [CCML07, Prop. 5.1], D is irreducible; since $D^2 = 2$, by Bertini's theorem it follows that D is smooth. Consider the system $|M| = |2B|$: the (set-theoretic) base locus of $|M|$ is contained in the (set-theoretic) base locus of $|B|$, which consists of 1 or 2 points. The restriction sequence $0 \rightarrow H^0(B) \rightarrow H^0(M) \rightarrow H^0(2K_D) \rightarrow 0$ is exact, since $H^1(B) = 0$; it follows that $|M|$ is free and, in the terminology of [CD89], it is a *superelliptic* system. By [CD89, Thm. 4.7.1], $M = 2B'$, where there are two possibilities for B' :

- (a) there exists elliptic half-pencils E_1, E_2 such that $E_1 E_2 = 1$ and $B' = E_1 + E_2$
- (b) there exists an elliptic half-pencil E and a nodal curve Z such that $EZ = 1$ and $B' = 2E + Z$

Since $2B = 2B' = M$, we either have $B = B'$ or $B = B' + K_Y$, and in either case B and B' are numerically equivalent. If case (a) occurs, then $(E_1 + E_2)B = 2$ and $(E_1 + E_2)C_i = 0$ for $i = 1, \dots, 5$. Since $|2E_i|$ is a free pencil for $i = 1, 2$ and $B^2 > 0$, it follows that $E_i B = 1$ and $E_i C_j = 0$ for $j = 1, \dots, 5$. So we have $E_i(2L) = E_i(B + C_1 + \dots + C_5) = 1$, a contradiction. So case (b) occurs. We claim that Z is one of the C_i . Assume by contradiction that this is not the case: then $(2E + Z)C_i = 0$ implies that Z is disjoint from the C_i . The divisor $C_1 + \dots + C_5 + Z \sim 2L - 2E$ has self-intersection -12 , hence $(L - E)^2 = -3$, contradicting the fact that the intersection form on $\text{NS}(Y)$ is even.

So Z is equal to, say, C_5 , and we have $B = 2E + C_5 + K_Y$, since $|2E + C_5|$ has C_5 as a fixed component while $|B|$ is an irreducible system.

(ii) follows immediately by (i). \square

Lemma 3.5. *Let S be a Godeaux surface with an involution σ of Enriques type. Then $\text{Tors}(S)$ is cyclic of order 4 and σ acts as the identity on $\text{Tors}(S)$.*

Proof. That $\text{Tors}(S)$ is cyclic of order 4 is proven in [CCML07, Prop. 5.3]. Here we describe explicitly $\text{Tors}(S)$. Since smooth blow ups do not change the torsion, we may replace S by X . Of course the element of order 2 is p^*K_Y . By Proposition 3.4 there is $N \in \text{Pic}(Y)$ such that $2N \equiv C_1 + \dots + C_4 + K_Y$; pulling back to X we obtain $2p^*N \equiv 2(\Gamma_1 + \dots + \Gamma_4) + p^*K_Y$, hence $p^*N - (\Gamma_1 + \dots + \Gamma_4)$ is a torsion element of order 4 and it is clearly σ -invariant. \square

Lemma 3.6. *Let S be a Godeaux surface with an involution σ of Enriques type, let $c: \tilde{S} \rightarrow S$ be the canonical cover and let $G = \text{Hom}(\text{Tors}(S), \mathbb{C}^*)$ be the Galois group of c . Then there is an involution $\tilde{\sigma}$ of \tilde{S} that lifts σ and commutes with G .*

Proof. Since the canonical cover is intrinsically associated with S , σ can be lifted to an automorphism h of \tilde{S} , so the point is to show that h can be taken to be an involution that commutes with G . We have

$$\tilde{S} = \text{Spec}(\oplus_{\eta \in \text{Tors}(S)} \eta),$$

and by Lemma 3.5 the action of G on $\oplus_{\eta \in \text{Tors}(S)} \eta$ preserves the summands. Thus h commutes with G . Denote by \tilde{G} the subgroup of $\text{Aut}(\tilde{S})$ generated by h and G : it is an abelian group of order 8 with a cyclic subgroup of order 4, hence it is either isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$ or to \mathbb{Z}_8 . To prove the lemma we have to exclude the latter possibility. Assume for contradiction that \tilde{G} is cyclic of order 8: then h generates \tilde{G} . So in particular h acts freely on \tilde{S} , because G does so. It follows that the group \tilde{G} acts freely on \tilde{S} , which is impossible, for instance because $K_{\tilde{S}}^2 = 4$ is not divisible by 8. \square

By definition, the canonical ring $R(\tilde{S})$ coincides with the paracanonical ring of S :

$$\oplus_{m \in \mathbb{N}, \eta \in \text{Tors}(S)} H^0(mK_S + \eta).$$

There are two possible choices of $\tilde{\sigma}$ as in Lemma 3.6; each of these choices induces a σ -linearization of the pluricanonical bundles $mK_S + \eta$ compatible with the multiplicative structure of $R(\tilde{S})$ and a \mathbb{Z}_2 -action on $H^0(mK_S + \eta)$ that lifts σ . So each vector space $H^0(mK_S + \eta)$ splits as a sum of two eigenspaces (corresponding to ± 1), whose dimensions we call the σ -type of $mK_S + \eta$.

We determine the σ -type in some cases:

Lemma 3.7. *Let S be a Godeaux surface and $\sigma \in \text{Aut}(S)$ an Enriques involution. Denoting by $1 \in \text{Tors}(S)$ a generator, the σ -type of $mK_S + i$, for $m = 1, 2, 4$ and $i \in \text{Tors}(S)$ is shown in row m , column i of Table 1.*

TABLE 1. σ -types of $mK_S + i$

$m \setminus i$	0	1	2	3
1	{0,0}	{1,0}	{1,0}	{1,0}
2	{2,0}	{1,1}	{2,0}	{1,1}
4	{5,2}	{4,3}	{5,2}	{4,3}

Proof. We may replace S by X , since this does not affect the σ -type. We recall the Hurwitz formula $K_X = p^*(K_Y + L)$, where as usual L is the line bundle such that $p_*\mathcal{O}_X = \mathcal{O}_Y \oplus L^{-1}$; in addition, by Lemma 3.5 and its proof, there is a line bundle $N \in \text{Pic}(Y)$ such that $2N \equiv K_Y + C_1 + \dots + C_4$ and our

chosen generator $1 \in \text{Tors}(X) \cong \text{Tors}(S)$ is equal to $p^*N - (\Gamma_1 + \cdots + \Gamma_4)$. So we have :

$$\begin{aligned}
 (3.3) \quad mK_X &\equiv p^*(mK_Y + mL) \\
 mK_X + 1 &\equiv p^*(mK_Y + mL + N) - (\Gamma_1 + \cdots + \Gamma_4) \\
 mK_X + 2 &\equiv p^*((m+1)K_Y + mL) \\
 mK_X + 3 &\equiv p^*((m+1)K_Y + mL + N) - (\Gamma_1 + \cdots + \Gamma_4).
 \end{aligned}$$

Recall also that $h^0(K_X + i) = 1$ for $i \neq 0$ and $h^0(mK_X + i) = 1 + \frac{m(m-1)}{2}$ for $m \geq 2$ and for every $i \in \text{Tors}(X)$. Using these remarks, the projection formulae for double covers and Kawamata-Viehweg vanishing, it is not hard to obtain Table 1.

As an example, consider $2K_X + 1$: using (3.3) and the relation $2L \equiv B + C_1 + \cdots + C_5$, gives $2K_X + 1 = p^*(B + N) + \Gamma_1 + \cdots + \Gamma_4 + 2\Gamma_5$. Since for $m > 0$ and for every i the fixed part of $|mK_X + i|$ contains $m(\Gamma_1 + \cdots + \Gamma_5)$, we have $2 = h^0(2K_X + 1) = h^0(p^*(B + N))$. The projection formula for double covers gives the following decomposition in \mathbb{Z}_2 -eigenspaces:

$$H^0(p^*(B + N)) = H^0(B + N) \oplus H^0(B + N - L).$$

We have $B + N \sim B + \frac{1}{2}(C_1 + \cdots + C_4)$: since B is nef and big, we may apply Kawamata-Viehweg vanishing and we obtain $h^0(B + N) = \chi(B + N) = 1$, and thus $2K_X + 1$ has σ -type $\{1, 1\}$. □

Conclusion of the proof of Theorem 3.2. We follow the steps of Reid's description of the paracanonical ring $R(S)$ taking into account also the action of the cyclic group G (of order 4). So in degree 1 we have generators $x_i \in H^0(K_S + i)$, $i = 1, 2, 3$ and in degree 2 we have two more generators $y_j \in H^0(2K_S + j)$, for $j = 1, 3$ and the element $g \in G$ acts on these generators as in Example 3.1. In addition, we may assume that all these generators are eigenvectors of $\tilde{\sigma}$, since $\tilde{\sigma}$ and g commute. Finally, up to replacing $\tilde{\sigma}$ by $\tilde{\sigma}g^2$, we may assume that y_1 is $\tilde{\sigma}$ invariant. The space $H^0(2K_S)$ is generated by x_2^2 and x_1x_3 : since by Lemma 3.7 the σ -type of $2K_S$ is $\{2, 0\}$, it follows that x_1 and x_3 are eigenvectors of $\tilde{\sigma}$ for the same eigenvalue. The space $H^0(2K_S + 1)$ is generated by x_2x_3 and y_1 and has type $\{1, 1\}$. It follows that x_2 and x_3 have opposite eigenvalues. Similarly, looking at $H^0(2K_S + 3)$ we conclude that y_3 is also $\tilde{\sigma}$ -invariant. So $\tilde{\sigma}$ has the form $(x_1, x_2, x_3, y_1, y_3) \mapsto (\pm x_1, \mp x_2, \pm x_3, y_1, y_3)$.

Now look at $H^0(4K_S)$: the two eigenspaces are spanned by

$$(3.4) \quad x_1^4, x_2^4, x_3^4, x_1^2x_3^2, x_1x_3x_2^2, y_1y_3$$

and by

$$x_1x_2y_1, x_2x_3y_3.$$

Since by Lemma 3.7 the σ -type of $4K_S$ is $\{5, 2\}$, there is a linear relation q_0 involving the monomials (3.4). The same argument shows the existence of

a relation q_2 between the monomials:

$$x_1^2 x_2^2, x_2^2 x_3^2, x_1^3 x_3, x_1 x_3^3, y_1^2, y_3^2.$$

Finally, we observe that the map $(x_1, x_2, x_3, y_1, y_3) \mapsto (-x_1, -x_2, -x_3, y_1, y_3)$ induces the identity on $\mathbb{P}(1, 1, 1, 2, 2)$, so $\tilde{\sigma}$ acts on $\mathbb{P}(1, 1, 1, 2, 2)$ as in Example 3.1. □

4. ENRIQUES SURFACES OF GODEAUX-QUOTIENT TYPE

Here we apply the results of the previous section to describe the Enriques surfaces that are (birational) quotients of a Godeaux surface by an involution.

We consider Enriques surfaces Y such that Y contains an elliptic half-pencil E and nodal curves C_1, \dots, C_5 such that:

- $EC_5 = 1, EC_1 = \dots = EC_4 = 0$
- $C_1 + \dots + C_4 + K_Y$ is divisible by 2 in $\text{Pic}(Y)$.

We call a surface Y as above an Enriques surface of *Godeaux-quotient type*. Proposition 3.4 has a converse:

Proposition 4.1. *In the above setting:*

- (i) *the system $|2E + C_5 + K_Y|$ is an irreducible pencil;*
- (ii) *let $B \in |2E + C_5 + K_Y|$ be a curve disjoint from C_1, \dots, C_5 ; then there exists a double cover $X \rightarrow Y$ branched on $B + C_1 + \dots + C_5$ and the minimal model of X is a Godeaux surface with an involution of Enriques type.*

Proof. The first assertion follows from the Riemann-Roch theorem and [CD89, Proposition 3.1.5]. For (ii) notice that $B + C_1 + \dots + C_5$ is even. Let $X \rightarrow Y$ be a double cover branched on $B + C_1 + \dots + C_5$. Standard double cover calculations (cf. for example [MLP04, Proposition 2.2]) yield the result. □

As a direct consequence of Propositions 4.1, 3.4 and Corollary 3.3, we have:

Corollary 4.2. *The Enriques surfaces of Godeaux-quotient type are an irreducible unirational subset of dimension 5 of the moduli space of Enriques surfaces.*

We now give an explicit construction of Enriques surfaces of Godeaux quotient type.

Example 4.3. Consider the quadric cone $\mathcal{Q} \subset \mathbb{P}^3$ defined by $y_0^2 - y_1 y_2 = 0$ and the involution τ of \mathcal{Q} defined by $[y_0, y_1, y_2, y_3] \mapsto [y_0, -y_1, -y_2, y_3]$. The linear system $|M|$ spanned by the invariant quadrics $y_0^2, y_1^2, y_2^2, y_3^2, y_0 y_3$ embeds the quotient surface \mathcal{Q}/τ in \mathbb{P}^4 as a quartic surface \mathcal{D} defined by $x_0^2 - x_1 x_2 = x_0 x_3 - x_4^2 = 0$. The surface \mathcal{D} (\mathcal{D}'_1 in the notation of [CD89, Ch.0, §4]) has two singular points of type A_1 at the points $P_1 = [0, 1, 0, 0, 0]$

and $P_2 = [0, 0, 1, 0, 0]$ (the “simple vertices”) and a singularity of type A_3 at the point $P_0 = [0, 0, 0, 1, 0]$ (the “ A_3 -vertex”).

An Enriques surface is called *special* if it contains a nodal curve C and an elliptic half-pencil E with $EC = 1$. All the special Enriques surfaces can be constructed as follows (cf. [Hor78a] and [Hor78b]).

Take an element B_0 in the linear system of τ -invariant quartic sections of \mathcal{Q} such that B_0 does not contain the fixed points of τ and has at most negligible singularities. The double cover $\tilde{Y} \rightarrow \mathcal{Q}$ is a $K3$ surface with canonical singularities. In particular it has two A_1 points over the vertex $[0, 0, 0, 1] \in \mathcal{Q}$. The involution τ can be lifted to a free involution $\tilde{\tau}$ of \tilde{Y} . The quotient surface $\tilde{Y}/\tilde{\tau}$ is an Enriques surface with canonical singularities, and by construction it is a double cover of \mathcal{D} branched over the singular points P_0, P_1, P_2 and on the image B of B_0 . The preimage of P_0 is an A_1 singular point, which gives a nodal curve C on the minimal resolution Y of $\tilde{Y}/\tilde{\tau}$; the preimage of the line joining P_0 and P_1 gives an elliptic half-pencil E of Y such that $EC = 1$.

We now specialize this construction in order to get an Enriques surface of Godeaux quotient type. We take $B_0 = D + \tau^*D$, where D is a general quadric section of \mathcal{Q} . The curve B_0 has 8 nodes at the intersection points of D and τ^*D , so in this case \tilde{Y} has 10 A_1 points, two occurring over the vertex of \mathcal{Q} and eight occurring over the nodes of B . These last eight points are an even set by Lemma 2.4. As in the proof of Lemma 2.4 consider the bidouble cover $h: Z \rightarrow \mathcal{Q}$ given by the relations $2L_1 \equiv D$, $2L_2 \equiv \tau^*D$, where $L_1 = L_2 = \mathcal{O}_{\mathcal{Q}}(1)$. As in §2.1 we denote by $G = \{1, g_1, g_2, g_3\}$ the Galois group of the bidouble cover and we assume that g_1 , respectively g_2 , fixes the preimage of D , respectively τ^*D , pointwise, so that $Z/g_3 = \tilde{Y}$. As explained in §2.2, it is possible to lift τ to an automorphism ρ of Z and the group $\tilde{G} < \text{Aut}(Z)$ generated by the Galois group $G \cong \mathbb{Z}_2^2$ and by ρ is isomorphic to the dihedral group D_4 . The subgroup $G < \tilde{G}$ contains two reflections conjugate to one another and the square of a rotation, so we may choose the lift ρ of τ to be a rotation. Since τ switches D and τ^*D , the action of ρ on G by conjugation switches g_1 and g_2 and fixes g_3 . It follows that g_1 and g_2 are reflections and $g_3 = \rho^2$. Now let $\tilde{\tau}$ be the automorphism of $\tilde{Y} = Z/\rho^2$ induced by ρ . The fixed locus of ρ^2 on Z is the set of 8 points lying over the nodes of $D + \tau^*D$. Since ρ acts freely on these points, it follows that ρ acts freely on Z and $\tilde{\tau}$ acts freely on \tilde{Y} (the fixed points of $\tilde{\tau}$ correspond to solutions $z \in Z$ of $\rho z = z$ or $\rho z = \rho^2 z$). Let Y be the minimal resolution of the surface $\tilde{Y}/\tilde{\tau} = Z/\rho$. The surface Y is a special Enriques surface that contains, besides $C_5 := C$ as in the general case, four additional disjoint nodal curves C_1, \dots, C_4 arising from the 4 nodes of $\tilde{Y}/\tilde{\tau}$ that are the images of the 8 nodes of \tilde{Y} . Since the nodes of \tilde{Y} are an even set, by Lemma 2.2 either $C_1 + \dots + C_4$ or $C_1 + \dots + C_4 + K_Y$ is even. Lemma 2.3 tells us that the latter case occurs, and therefore Y is an Enriques surface of Godeaux-quotient type.

Theorem 4.4. *The general Enriques surface of Godeaux-quotient type can be constructed as in Example 4.3.*

Proof. Since $\text{Aut}(\mathcal{Q})$ has dimension 3, the construction gives a 5-dimensional family of Enriques surfaces of Godeaux-quotient type and the statement follows by Corollary 4.2. \square

5. A CONSTRUCTION OF THE GENERAL GODEAUX SURFACE WITH AN ENRIQUES INVOLUTION

We give an alternative description of the general Godeaux surface with an involution of Enriques type, that will be used in §6 to compute some stable degenerations.

We keep the notation of the previous section (especially of Example 4.3). We take B_1 a general quadratic section of \mathcal{Q} , $B_2 = \tau^* B_1$ and B_3 a general hyperplane section containing the two smooth fixed points Q_1 and Q_2 of τ (notice that B_3 is τ -invariant). Consider the minimal resolution $\mathbb{F}_2 \rightarrow \mathcal{Q}$, denote by Γ the exceptional curve and use the same letter to denote curves on \mathcal{Q} and their pull-backs to \mathbb{F}_2 . By §2.1 there exists a bidouble cover $T_0 \rightarrow \mathbb{F}_2$ with branch divisors $B_1, B_2, B_3 + \Gamma$ and by §2.2 the involution of \mathbb{F}_2 induced by τ can be lifted to an automorphism of T_0 . The preimage of Γ is the disjoint union of two irreducible -1 -curves. Contracting these two curves, one obtains a bidouble cover $q: T \rightarrow \mathcal{Q}$, with T smooth, with branch divisors B_1, B_2 and B_3 , which is branched also on the vertex $Q_0 = [0, 0, 0, 1]$ of \mathcal{Q} . By the Hurwitz formula, one has $K_T \sim \frac{1}{2}B_3$, hence T is smooth minimal of general type with $K_T^2 = 2$. The group $\tilde{G} < \text{Aut}(T)$ generated by the Galois group $G = \{1, g_1, g_2, g_3\} \cong \mathbb{Z}_2^2$ of q and by a lift of τ is isomorphic to D_4 (cf. §2.2). Denote by $\rho \in D_4$ an element of order 4: then ρ is a lift of τ , ρ^2 is an element of G and commutes with ρ . Since τ exchanges B_1 and B_2 , we have $g_3 = \rho^2$ and g_1 and $g_2 = g_1\rho^2$ are reflections. As in Example 4.3, the surface $\tilde{Y} := T/\rho^2$ is a $K3$ surface with 10 nodes.

Lemma 5.1. *In the above setting:*

- (i) $g_1\rho$ and $g_1\rho^3$ induce a fixed point free involution of \tilde{Y} ;
- (ii) the surfaces $T/g_1\rho$ and $T/g_1\rho^3$ are Godeaux surfaces with an Enriques involution.

Proof. (i) There are two liftings of τ to \tilde{Y} , one induced by ρ and the other one induced by $g_1\rho$. We know (cf. Example 4.3) that one of these acts freely, while the other one fixes 8 points. Assume for contradiction that ρ induces a fixed point free involution $\tilde{\tau}$ and denote by Y the minimal desingularization of $\tilde{Y}/\tilde{\tau}$. By Example 4.3, Y is an Enriques surface of Godeaux quotient type; in particular $B + C_1 + \dots + C_5$ is divisible by 2 in $\text{Pic}(Y)$, where we denote by B the strict transform of the image of B_3 and by C_1, \dots, C_5 the nodal curves that arise from the resolution of the images of the 10 nodes of \tilde{Y} . On the other hand, arguing as we did at the end of Example 4.3 we see

that $B + C_1 + \cdots + C_5 + K_Y$ is divisible by 2 in $\text{Pic}(Y)$. It follows that K_Y is divisible by 2 in $\text{Pic}(Y)$, a contradiction. So the fixed point free involution $\tilde{\tau}$ of \tilde{Y} that lifts τ is induced by $g_1\rho$. Clearly, also $g_1\rho^3$ induces the same involution.

(ii) By (i) $g_1\rho$ is a fixed point free involution of \tilde{Y} and the same is true of the conjugate involution $g_1\rho^3$. The surfaces $S_1 := T/g_1\rho$ and $S_2 := T/g_1\rho^3$ are isomorphic; they are smooth minimal of general type with $K_{S_i}^2 = 1$ for $i = 1, 2$, hence they are Godeaux surfaces. The involution ρ^2 induces on S_1 and S_2 an Enriques involution with quotient $\tilde{Y}/\tilde{\sigma}$. \square

Proposition 5.2. *The family of surfaces constructed as in Lemma 5.1, (ii) contains a dense open subset of the family of Godeaux surfaces with an Enriques involution.*

Proof. By Corollary 3.3, it suffices to count dimensions. \square

6. STABLE DEGENERATIONS OF GODEAUX SURFACES WITH AN ENRIQUES INVOLUTION

At the beginning of this section we recall some facts on stable Godeaux surfaces. Then we describe some examples, obtained by letting the branch divisors in the construction given in §5 of the general Godeaux surfaces with an Enriques involution acquire singularities or multiple components.

6.1. Non-normal Gorenstein stable Godeaux surfaces. The notion of stable surface generalizes that of (canonical model of) minimal surface of general type in the same way as the notion of stable curve generalizes that of smooth curve of genus > 1 : there exists a projective coarse moduli space $\overline{\mathcal{M}}_{a,b}$ parametrizing stable surfaces with fixed numerical invariants $K^2 = a$ and $\chi = b$ and the moduli space of surfaces of general type with the same invariants is an open subset $\mathcal{M}_{a,b} \subset \overline{\mathcal{M}}_{a,b}$ (cf. [Ale06] for an exposition of the theory of stable varieties and, more generally, of stable pairs).

We recall the definition: a *stable surface* is a projective surface S such that:

- in the terminology of [Kol13] the surface S is *demi-normal*. This means that S satisfies condition S_2 of Serre and there exists an open subset $S_0 \subset S$ such that $S \setminus S_0$ is a finite set and for every $x \in S_0$ the point x is either smooth or double crossings (i.e., S is locally isomorphic to $xy = 0$ in the analytic or étale topology).
- let $\bar{S} \rightarrow S$ be the normalization map and let $\bar{D} \subset \bar{S}$ be the *double locus*, that is, \bar{D} the effective divisor defined by the conductor ideal sheaf; then (\bar{S}, \bar{D}) is a log-canonical pair.
- there exists an integer m such that $\mathcal{O}_S(mK_S)$ is an ample line bundle.

If S is a stable surface, we denote by $\nu(S)$ the *Cartier index* of S , namely the smallest $m > 0$ such that mK_S is Cartier.

We call a stable surface with $K_S^2 = \chi(S) = 1$ a *stable Godeaux surface*; we say that S is *classical* if it has at most rational double points, i.e., if it is the canonical model of a minimal smooth surface of general type Y with $K_Y^2 = \chi(Y) = 1$. We are mainly interested in the case in which S is Gorenstein. Under this assumption, one has $h^1(\mathcal{O}_S) = h^2(\mathcal{O}_S) = 0$ ([FPR14a, Prop. 4.2]) and the possibilities for the pair (\bar{S}, \bar{D}) associated to a non-classical Godeaux surface S are quite restricted:

Theorem 6.1 ([FPR14b], Thm. 3.7 and 4.1). *Let S be a non-classical stable Godeaux surface and let (\bar{S}, \bar{D}) be the corresponding log-canonical pair. If S is Gorenstein, then one of the following cases occurs:*

- (N) $S = \bar{S}$, namely S is normal. Denote by $\epsilon: \tilde{S} \rightarrow S$ the minimal desingularization; in this case $\chi(\tilde{S}) = 0$ and the only non canonical singularity of S is an elliptic singularity;
- (P) $\bar{S} = \mathbb{P}^2$, \bar{D} a quartic;
- (dP) \bar{S} is a del Pezzo surface of degree 1, with at most canonical singularities, and $D \in |-2K_{\bar{S}}|$;
- (E₊) \bar{S} is the symmetric product of a curve E of genus 1 and \bar{D} is a stable curve of genus 2 which is a trisection of the Albanese map $\bar{S} \rightarrow E$.

Remark 1. More precisely, in [FPR] it is shown that in case (N) the surface \tilde{S} is either the blow up of a bielliptic surface at a point or a surface ruled over an elliptic curve and the bielliptic case is completely classified. An example with \tilde{S} ruled appears in [Lee00, Ex. 2.14]; in §6.2 we give a new one.

The non-normal stable Gorenstein Godeaux surfaces of type (dP) are described in [Rol14], where it is shown that they form an irreducible component of the moduli space, hence in particular they are not smoothable.

The non-normal stable Gorenstein Godeaux surfaces of type (P) and (E₊) are classified in [FPR].

Here we recall the description of one family of surfaces of type (P) such that the general surface in the family has an involution. These surfaces are obtained in §6.2 as specializations of the Godeaux surfaces with an Enriques involution, and therefore they are smoothable (cf. Proposition 6.4).

Example 6.2. Let $P_1, \dots, P_4 \in \mathbb{P}^2$ be independent points and let $\phi: \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be the projective automorphism such that $\phi(P_i) = P_{i+1}$ for $1 \leq i \leq 4$ (indices are taken modulo 4). The automorphism ϕ induces on the pencil \mathcal{F} of conics through P_1, \dots, P_4 an involution that fixes the reducible conic $L(P_1, P_3) + L(P_2, P_4)$ and a smooth conic $C_0 \in \mathcal{F}$. We take $\bar{S} = \mathbb{P}^2$ and $\bar{D} = C + \phi_*C$, where $C \in \mathcal{F} \setminus \{C_0\}$ is a smooth conic. By [Kol13, Thm. 5.13] (cf. also [FPR14b, Thm. 3.2] for the Gorenstein condition) in order to construct a Gorenstein stable surface with $K^2 = 1$ with normalization (\bar{S}, \bar{D}) , one has to give an involution ι of the normalization $C \sqcup \phi_*C$ of \bar{D} with the property that ι acts freely on the eight preimages of P_1, \dots, P_4 . We take ι to be the involution that exchanges C and ϕ_*C and identifies C with ϕ_*C via ϕ . One

has $\chi(S) = 1$ by [FPR14b, Prop. 3.4]. The involution ϕ^2 of \mathbb{P}^2 commutes with ι and therefore it induces an involution of S (cf. [FPR14a, §3.B]).

6.2. Degenerating Godeaux surfaces with an Enriques involution.

A way of obtaining stable degenerations of a Godeaux surface with an Enriques involution is to apply the construction described in §5 relaxing the assumption that the branch divisors be general. Keeping the notation of §5, we take B_1 a divisor in $|\mathcal{O}_{\mathcal{Q}}(2)|$, $B_2 = \tau^*B_1$, B_3 a hyperplane section through Q_1 and Q_2 such that the pair $(\mathcal{Q}, \frac{1}{2}(B_1 + B_2 + B_3))$ is log-canonical and we construct the bidouble cover $T \rightarrow \mathcal{Q}$ with branch data B_1, B_2, B_3 . Observe that ρ induces an isomorphism between the quotient surfaces $T/g_1\rho$ and $T/g_1\rho^3$; we abuse notation and refer to either of these surfaces as to S . By Proposition 5.2 the surface S is a degeneration of the general Godeaux surfaces with an Enriques involution. The next result shows that it is indeed a stable degeneration:

Lemma 6.3. *Consider the setup and notation of §5 and assume that the pair $(\mathcal{Q}, \frac{1}{2}(B_1 + B_2 + B_3))$ is log-canonical.*

Then:

- (i) *T is a stable Godeaux surface with $\nu(T) = 1$ or 2. If $Q_0 \notin B_1 + B_2 + B_3$ and $B_1 \cap B_2 \cap B_3 = \emptyset$, then T is Gorenstein.*
- (ii) *S is a stable Godeaux surface such that $\nu(S)$ divides $2\nu(T)$;*
- (iii) *if $B_1 + B_2$ does not contain any of the fixed points Q_0, Q_1, Q_2 of τ on \mathcal{Q} , then $T \rightarrow S$ is an étale morphism, and in particular $\nu(S) = \nu(T)$.*

Proof. (i) The cover $T \rightarrow \mathcal{Q}$ is demi-normal by [AP12, Thm. 1.9]. By Prop. 2.5, ibidem, the surface T is slc and $2K_T$ is the pull back of $2K_{\mathcal{Q}} + (B_1 + B_2 + B_3) = H$, where H is the hyperplane section of \mathcal{Q} . Hence K_T is ample and 2-Cartier.

If $Q_0 \notin B_1 + B_2 + B_3$, then T is smooth (hence Gorenstein) over Q_0 ; if $B_1 \cap B_2 \cap B_3 = \emptyset$ then locally over every smooth point of \mathcal{Q} , $T \rightarrow \mathcal{Q}$ is the composition of two flat double covers and therefore it is Gorenstein.

(ii) Since $g_1\rho$ lifts τ , that has only isolated fixed points, the quotient map $T \rightarrow S$ is unramified in codimension 1, hence again by [AP12, Prop. 2.5] we have that S is an slc surface and K_S is ample, since it pulls back to K_T . In addition, the argument in the proof of [AP12, Lem. 2.3] shows that $\nu(S)$ divides $2\nu(T)$. The fact that $K_S^2 = \chi(\mathcal{O}_S) = 1$ follows from the fact that S can be obtained as a flat limit of smooth Godeaux surfaces and so S is a stable Godeaux surface.

(iii) It is enough to show that the involution of $Y := S/\rho^2$ induced by $g_1\rho$ is base point free. If B_1 and B_2 are general, Y is a nodal $K3$ surface and the involution induced by ρ fixes all the preimages of Q_0, Q_1 and Q_2 (cf. proof of Lemma 5.1). By continuity, the involution induced by ρ fixes the preimages of the fixed points of τ for every choice of B_1 and B_2 . Since g_1 induces the covering involution of $Y \rightarrow \mathcal{Q}$, if $Y \rightarrow \mathcal{Q}$ is unramified over Q_0, Q_1, Q_2 ,

then the involution of Y induced by $g_1\rho$ acts freely on the preimages of Q_0, Q_1, Q_2 , hence it acts freely on Y . \square

6.3. Examples of degenerations. We examine now some instances of the situation of §6.2. Recall that T (and S) is normal iff $B_1 + B_2 + B_3$ is a reduced divisor; in general, the normalization \bar{T} of T is a bidouble cover of \mathcal{Q} whose construction is described in [Par91, §3]. For the description of the possible singularities of T we refer the reader to [AP12, §3].

(1) B_1 and B_2 intersect at two points R_1, R_2 that are double points of both.

An example of this type can be constructed as follows. Choose $R_1 \in \mathcal{Q}$ general and set $R_2 = \tau(R_1)$. If H_1, \dots, H_4 are general hyperplane sections containing R_1 and R_2 , then $H_1 + H_2$ and $H_3 + H_4$ span a pencil of quadric sections. We take B_1 a general element of this pencil, so that B_1 has ordinary double points at R_1 and R_2 and is smooth elsewhere; as usual, we set $B_2 = \tau^*B_1$. Since $B_1B_2 = 8$, the divisor $B_1 + B_2$ has ordinary quadruple points at R_1 and R_2 . We assume that B_3 is general; by Lemma 6.3, T and S are both Gorenstein. The surface T has two elliptic singularities U_1 and U_2 of degree 4 over R_1 and R_2 (cf. Table 1 of [AP12, §3]). These singularities map in S to one elliptic singularity of the same type, hence $1 = \chi(S) = \chi(\tilde{S}) + 1$, where \tilde{S} is the minimal desingularization of S . The minimal desingularization $\tilde{T} \rightarrow T$ is obtained by blowing up \mathcal{Q} at R_1 and R_2 and taking base change and normalization; the exceptional curves of the blow-up $\hat{\mathcal{Q}} \rightarrow \mathcal{Q}$ are not contained in the branch locus of $\tilde{T} \rightarrow \hat{\mathcal{Q}}$. Therefore the strict transforms on $\hat{\mathcal{Q}}$ of the plane sections of \mathcal{Q} through R_1 and R_2 meet the branch locus of $\tilde{T} \rightarrow \hat{\mathcal{Q}}$ only at two points, and so their preimages in T are pairs of rational curves. So T is ruled and therefore S and \tilde{S} are ruled, too. Since $\chi(\tilde{S}) = 0$, the surface \tilde{S} is ruled over an elliptic curve.

This is a new example of case (N) of Theorem 6.1 with S ruled; the other known example (cf. [Lee00, Ex. 2.14]) has an elliptic singularity of degree 3.

(2) $B_1 = 2H$, with H a general hyperplane section.

We have $B_2 = 2\tau^*H$ and we take B_3 general; by Lemma 6.3, T and S are both Gorenstein. In this case, the surface $\tilde{Y} = T/\rho^2$ is the union of two copies of \mathcal{Q} glued along the curve $H + \tau^*H$. The surface T is non-normal and has two irreducible components, both isomorphic to the double cover of \mathcal{Q} branched on the plane section B_3 and on the vertex Q_0 of \mathcal{Q} , and therefore both isomorphic to \mathbb{P}^2 . By Lemma 6.3, the surface S is Gorenstein and therefore irreducible, since $K_S^2 = 1$. So $g_1\rho$ permutes the two components of T and the normalization \bar{S} of S is isomorphic to \mathbb{P}^2 , namely S is as in case (P) of Theorem 6.1. The surface $\bar{S} = \mathbb{P}^2$ can be naturally identified with one of the irreducible component of T ; we denote by $\pi: \bar{S} \rightarrow \mathcal{Q}$ the degree

2 map induced by this identification. The double locus $\bar{D} \subset \bar{S}$ is the union of two conics, $C_1 := \pi^*H$ and $C_2 := \pi^*(\tau^*H)$, that are identified with one another by the involution ι of $C_1 \sqcup C_2$ induced by the map $\bar{S} \rightarrow S$.

We claim that the surface S belongs to the family constructed in Example 6.2. Let R_1, R_2 be the intersection points of H and τ^*H in \mathcal{Q} and write $\pi^{-1}(R_1) = \{P_1, P_3\}$ and $\pi^{-1}(R_2) = \{P_2, P_4\}$. The points P_1, \dots, P_4 are the base points of the pencil of conics spanned by C_1 and C_2 . By construction, the involution ι of $C_1 \sqcup C_2$ lifts the involution of $H + \tau^*H$ given by τ . The involution τ lifts to an automorphism of \bar{S} that exchanges the sets $\pi^{-1}(R_1)$ and $\pi^{-1}(R_2)$ and exchanges the conics C_1 and C_2 . Elementary arguments on pencils of plane conics show that such a map is either the that automorphism ϕ that induces a cyclic permutation of P_1, \dots, P_4 or its inverse ϕ^3 . So, possibly up to relabelling the P_i , the involution ι of $C_1 \sqcup C_2$ induced by the normalization map $\bar{S} \rightarrow S$ switches C_1 and C_2 and identifies C_1 with C_2 via ϕ . Since letting H vary in the pencil of plane sections through R_1 and R_2 we can obtain any conic in the pencil spanned by C_1 and C_2 , we have proven the following:

Proposition 6.4. *The surfaces in the family of Example 6.2 are smoothable.*

(3) B_1 and B_2 have a common component which is a hyperplane section. Take $B_1 = H_0 + H_1$, where H_0 is a τ -invariant hyperplane section and H_1 is a general one, so that $B_2 = H_0 + \tau^*H_1$, and take B_3 general. Assume that H_0 does not contain the vertex Q_0 of \mathcal{Q} , hence H_0 contains the two smooth fixed points Q_1 and Q_2 of τ . By Table 2 of [AP12, §3] the singularities of T over Q_1 and Q_2 are not Gorenstein, so $\nu(T) = 2$ by Lemma 6.3, and it follows that S is not Gorenstein either.

By [Par91, §3], the normalization \bar{T} of T is the bidouble cover of \mathcal{Q} branched on $H_1, \tau^*H_1, B_3 + H_0$ and the vertex Q_0 of \mathcal{Q} . The surface \bar{T} has a pair of singular points of type A_1 over Q_1 and over Q_2 and is smooth elsewhere. By the Hurwitz formula the canonical class $K_{\bar{T}}$ is numerically equivalent to 0. Taking base change of $\bar{T} \rightarrow \mathcal{Q}$ with the minimal resolution $\mathbb{F}_2 \rightarrow \mathcal{Q}$ one obtains a flat bidouble cover $T_0 \rightarrow \mathbb{F}_2$. The standard formulae for double covers give $p_g(T_0) = q(T_0) = 0$, hence \bar{T} is an Enriques surface with four nodes. The involution $g_1\rho$ of T induces an involution of the minimal desingularization \tilde{T} of T , whose fixed locus is contained in the preimages of the points $Q_0, Q_1, Q_2 \in \mathcal{Q}$. The preimage of Q_0 consists of two smooth points, while the preimage of $\{Q_1, Q_2\}$ is the disjoint union of four nodal curves. Assume that one of these nodal curves is preserved by $g_1\rho$; then a local computation shows that this curve is not fixed pointwise by $g_1\rho$. Summing up, the fixed locus of $g_1\rho$ on \tilde{T} is finite. It follows that the quotient surface $\tilde{T}/g_1\rho$ is again an Enriques surface, and so is \bar{S} , since it is birational to $\tilde{T}/g_1\rho$.

This example shows that if we remove the assumption that S is Gorenstein, then Theorem 6.1 does not hold any more.

(4) $B_1 = H_1 + 2F_1$, where H_1 is a general hyperplane section and F_1 is a general ruling of \mathcal{Q} .

Set $H_2 = \tau^*H_1$, $F_2 = \tau^*F_1$, so that $B_2 = H_2 + 2F_2$. The surface T is singular above F_1 and F_2 . The normalization \bar{T} of T is a bidouble cover of \mathcal{Q} branched on the three hyperplane sections H_1 , H_2 and B_3 , so $K_{\bar{T}}$ is numerically equivalent to the pull-back of $-\frac{1}{2}H_1$, and \bar{T} is a del Pezzo surface of degree 2. The map $\bar{T} \rightarrow \mathcal{Q}$ is unramified over the vertex Q_0 , hence the singularities of \bar{T} are four points U_1, U_2, U_3, U_4 of type A_1 occurring above Q_0 . The elements $g_1, g_2 = g_1\rho^2, \rho^2$ of D_4 act on U_1, U_2, U_3, U_4 switching them in pairs, so ρ acts as a cyclic permutation of order 4 and $g_1\rho$ switches, say, U_1 and U_3 and fixes U_2 and U_4 . Looking at the minimal resolution \tilde{T} of \bar{T} , one sees that $g_1\rho$ has two isolated fixed points on each of the nodal curves corresponding to U_2 and U_4 , hence the fixed locus of $g_1\rho$ on \tilde{T} is a finite set and the quotient surface $\bar{S} = \tilde{T}/g_1\rho$ has canonical singularities (the images of U_2 and U_4 are points of type A_3). Hence \bar{S} is a del Pezzo surface of degree 1.

The double locus $D_T \subset \bar{T}$ is the preimage of $F_1 + F_2$: it consists of two smooth rational curves Γ_1 and Γ_2 meeting transversely at U_1, \dots, U_4 and it is an antibicanonical curve. The double locus $\bar{D} \subset \bar{S}$ is the image of D_T : it is an irreducible curve with $p_a = 1$, since it is smooth at the images of U_2 and U_4 and it has a node at the image point of U_1 and U_3 . The curve \bar{D} is numerically equivalent to an antibicanonical curve, since it pulls back to D_T , but it is not Cartier since it is smooth at the A_3 points of \bar{S} (notice also the failure of the usual adjunction formula), hence it is not in $|-2K_{\bar{S}}|$. So this case is different from case (dP) of Theorem 6.1. In fact, the surface S is not Gorenstein, since $K_{\bar{S}} + \bar{D}$ is not Cartier.

REFERENCES

- [Ale06] Valery Alexeev. Higher-dimensional analogues of stable curves. *Proceedings of Madrid ICM2006*, 352(2):515–536, 2006.
- [AP12] Valery Alexeev and Rita Pardini. Non-normal abelian covers. *Compos. Math.*, 148(4):1051–1084, 2012.
- [Bar84] Rebecca Barlow. Some new surfaces with $p_g = 0$. *Duke Math. J.*, 51(4):889–904, 1984.
- [Bar85] Rebecca Barlow. A simply connected surface of general type with $p_g = 0$. *Invent. Math.*, 79(2):293–301, 1985.
- [CCML07] Alberto Calabri, Ciro Ciliberto, and Margarida Mendes Lopes. Numerical Godeaux surfaces with an involution. *Trans. Amer. Math. Soc.*, 359(4):1605–1632 (electronic), 2007.
- [CD89] François R. Cossec and Igor V. Dolgachev. *Enriques surfaces. I*, volume 76 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1989.
- [FPR] Marco Franciosi, Rita Pardini, and Sönke Rollenske. Stable Gorenstein Godeaux surfaces. in preparation.
- [FPR14a] Marco Franciosi, Rita Pardini, and Sönke Rollenske. Computing invariants of semi-log-canonical surfaces. preprint arXiv:1404.3548, 2014.

- [FPR14b] Marco Franciosi, Rita Pardini, and Sönke Rollenske. Log-canonical pairs and Gorenstein stable surfaces with $K_X^2 = 1$. preprint arXiv:1403.2159, to appear in *Compositio Mathematica*, 2014.
- [Hor78a] Eiji Horikawa. On the periods of Enriques surfaces. I. *Math. Ann.*, 234(1):73–88, 1978.
- [Hor78b] Eiji Horikawa. On the periods of Enriques surfaces. II. *Math. Ann.*, 235(3):217–246, 1978.
- [KL00] Jonghae Keum and Yongnam Lee. Fixed locus of an involution acting on a Godeaux surface. *Math. Proc. Cambridge Philos. Soc.*, 129(2):205–216, 2000.
- [Kol13] János Kollár. *Singularities of the minimal model program*, volume 200 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács.
- [Lee00] Yongnam Lee. Semistable degeneration of Godeaux surfaces with relatively nef canonical bundle. *Math. Nachr.*, 219:135–146, 2000.
- [LP07] Yongnam Lee and Jongil Park. A simply connected surface of general type with $p_g = 0$ and $K^2 = 2$. *Invent. Math.*, 170(3):483–505, 2007.
- [Mag] S. Maggiori. Automorphisms of numerical Godeaux surfaces with torsion of order 3, 4, or 5. arXiv:1002.3494.
- [Miy75] Yoichi Miyaoka. Tricanonical map of a certain class of surfaces. *Sûrikaiseikikenkyûsho Kôkyûroku*, (240):31–47, 1975. A Symposium on Complex Manifolds (Kyoto, 1974).
- [MLP04] Margarida Mendes Lopes and Rita Pardini. A new family of surfaces with $p_g = 0$ and $K^2 = 3$. *Ann. Sci. École Norm. Sup. (4)*, 37(4):507–531, 2004.
- [Par91] Rita Pardini. Abelian covers of algebraic varieties. *J. Reine Angew. Math.*, 417:191–213, 1991.
- [PPS13] Heesang Park, Jongil Park, and Dongsoo Shin. Surfaces of general type with $p_g = 1$ and $q = 0$. *J. Korean Math. Soc.*, 50(3):493–507, 2013.
- [Rei78] Miles Reid. Surfaces with $p_g = 0$, $K^2 = 1$. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 25(1):75–92, 1978.
- [Rol14] Sönke Rollenske. A new irreducible component of the moduli space of stable Godeaux surfaces. preprint arXiv:1404.7027, 2014.

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